

CHAPTER 21: THE NATURAL LOGARITHM AND EXPONENTIAL

1. THE NATURAL LOGARITHM

Recall that the power rule for integration does not answer the question:

$$\int \frac{1}{x} dx = ?$$

However, the Fundamental Theorem of Calculus (Part 1) tells us how to construct an antiderivative of $1/x$; namely, for any $a \in (0, \infty)$, the function

$$F(x) = \int_a^x \frac{1}{t} dt$$

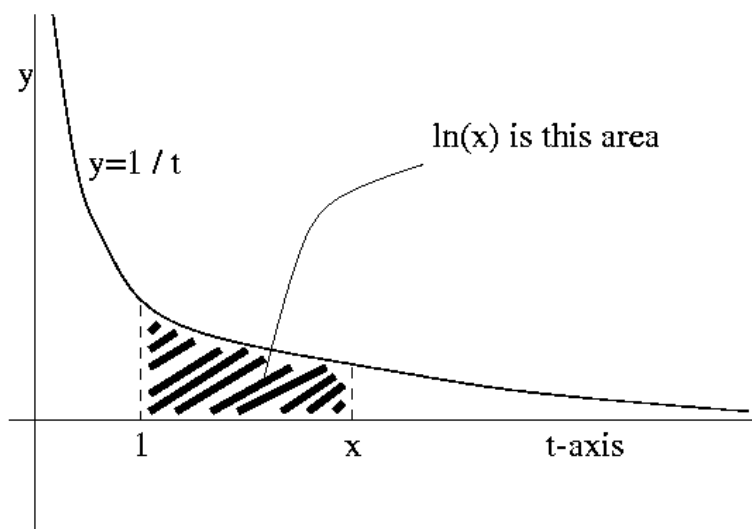
will do the trick. When we use the value $a = 1$, the resulting function is one of the most important in mathematics and its applications and – for reasons that will only become clear later – is called the *natural logarithm*.

Definition 1.1. The *natural logarithm function* is the function

$$\ln : (0, \infty) \rightarrow \mathbb{R}$$

defined by the formula

$$\ln(x) = \int_1^x \frac{1}{t} dt \quad \text{for } x > 0$$



To spell it out: the natural logarithm of the number x is the area under the curve $y = 1/t$ from $t = 1$ to $t = x$.

This is a complicated way to define a function (but there is no easier way to find an anti-derivative of $1/x$), and it is not easy to evaluate the function at any number except $x = 1$.

What *do* we know about this new function $\ln(x)$?

To begin with, we know its derivative: By the Fundamental Theorem (Part 1),

$$\frac{d}{dx} \ln(x) = \frac{1}{x} \quad \text{for } x > 0$$

Furthermore, it is easy to evaluate at 1:

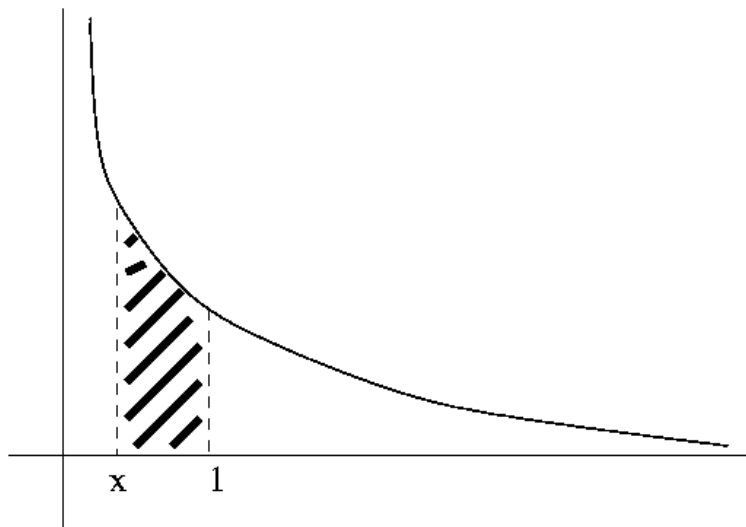
$$\ln(1) = \int_1^1 \frac{1}{t} dt = 0$$

Observe also that

$$\ln(x) > 0 \quad \text{if } x > 1$$

since, for every $x > 1$, $\ln(x)$ is the area of a region above the x -axis. However,

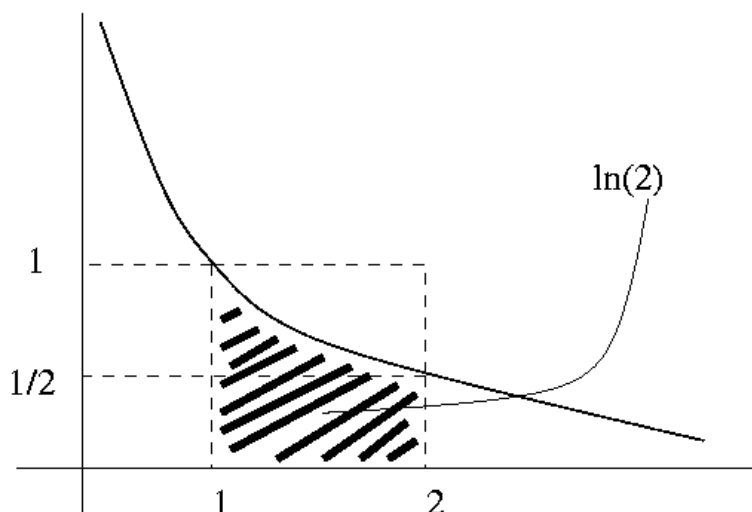
$$\ln(x) < 0 \quad \text{if } x < 1 :$$



$$\ln(x) = \int_1^x \frac{1}{t} dt = - \int_x^1 \frac{1}{t} dt = -(\text{shaded area})$$

For any value other than $x = 1$, $\ln(x)$ is hard to evaluate (without using a calculator). However, it is easy enough to get an estimate of the size of $\ln(a)$ for any given number a .

For instance, here is a picture of $\ln(2)$:



From the diagram, (comparing the shaded area with inscribed and circumscribed rectangles)

$$\frac{1}{2} < \ln(2) < 1$$

In fact (use your calculator)

$$\ln(2) \approx 0.693\dots$$

$$\ln(3) \approx 1.0986\dots$$

2. ALGEBRAIC PROPERTIES OF THE NATURAL LOGARITHM

The first clue as to why the function $\ln(x)$ is called a ‘logarithm’ comes from the following theorem, which shows that this mysterious new function has exactly the properties you would expect a logarithm function to have:

- Theorem 2.1.** (1) $\ln(1) = 0$
 (2) $\ln(ax) = \ln(a) + \ln(x)$ if $a, x > 0$
 (3) $\ln\left(\frac{a}{x}\right) = \ln(a) - \ln(x)$
 (4) $\ln\left(\frac{1}{x}\right) = -\ln(x)$
 (5) $\ln(x^a) = a \ln(x)$ for $x > 0, a \in \mathbb{R}$

Proof:

- (1) Already noted above.
 (2) Let $F(x) = \ln(ax)$, $G(x) = \ln(a) + \ln(x)$. We wish to show that $F(x) = G(x)$ for all $x > 0$. To begin with, we show that they have the same derivative.

$$G'(x) = 0 + \frac{1}{x} = \frac{1}{x}$$

$$F'(x) = \frac{1}{ax} \cdot a = \frac{1}{x}$$

Thus $F'(x) = G'(x)$ for all x and therefore $F(x) = G(x) + C$, for some number C .

To find the value of C , let $x = 1$

$$\ln(a) = F(1) = G(1) + C = \ln(a) + C \implies C = 0$$

So $C = 0$ and therefore $F(x) = G(x)$ for all $x > 0$, which is what we wanted to demonstrate.

(3) For any $a, x > 0$

$$\begin{aligned} \ln\left(\frac{a}{x}\right) + \ln(x) &= \ln\left(\frac{a}{x} \cdot x\right) \quad (\text{by (2)}) \\ &= \ln(a) \end{aligned}$$

(4) Take $a = 1$ in (3) and use the fact that $\ln(1) = 0$:

$$\ln\left(\frac{1}{x}\right) = \ln(1) - \ln(x) = -\ln(x).$$

(5) Let $F(x) = \ln(x^a)$, $G(x) = a \ln(x)$ and show that $F'(x) = G'(x)$ by the same method as in the proof of (3):

$$\begin{aligned} F'(x) &= \frac{1}{x^a} \cdot ax^{a-1} = \frac{a}{x} \\ G'(x) &= a \cdot \frac{1}{x} = \frac{a}{x} \end{aligned}$$

So $F(x) = G(x) + C$ for some C . Let $x = 1$:

$$0 = F(1) = G(1) + C = 0 + C \implies C = 0$$

So $F(x) = G(x)$.

3. THE NATURAL LOGARITHM IS A LOGARITHM

Thus $\ln(x)$ has all the properties we expect of a logarithm function.

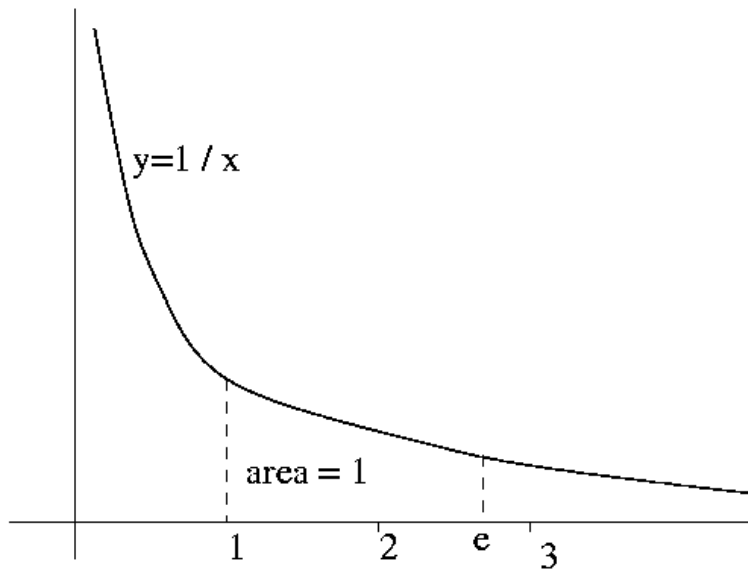
Is it *really* a logarithm function or does it just happen to have similar properties?

If it *is* a logarithm function then it must be a logarithm to some *base*. How do you determine the base of a logarithm function if you are not given it to start with?

Observe that the logarithm to the base b satisfies $\log_b(b) = 1$; i.e. for any logarithm the base is that number at which the logarithm takes the value 1.

Since $\ln(2) < 1$ and $\ln(3) > 1$, there is (by the Intermediate Value Theorem, since $\ln x$ is certainly continuous – why?) a number, traditionally denoted e , between 2 and 3 satisfying

$$\ln(e) = 1$$



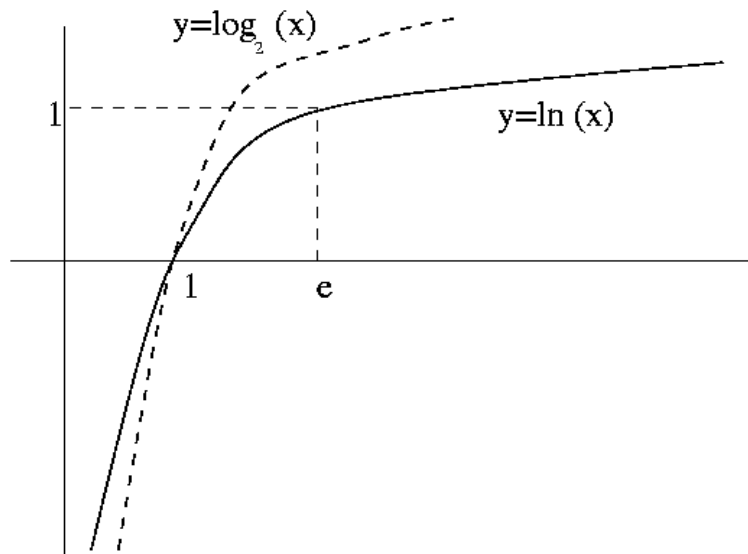
In fact $e \approx 2.718281828459\dots$ (it is irrational, however). Thus, if the natural logarithm really is a logarithm, it can only be the logarithm to the base e . The next theorem establishes that this is indeed the case:

Theorem 3.1.

$$\ln(x) = \log_e(x) \quad \text{for all } x > 0$$

Proof: For $x > 0$, $x = e^{\log_e(x)}$. So

$$\begin{aligned} \ln(x) &= \ln(e^{\log_e(x)}) \\ &= \log_e(x) \cdot \ln(e) \quad (\text{by (5)}) \\ &= \log_e(x) \end{aligned}$$



Observe that it would not have been possible to *define* the natural logarithm to be the logarithm to the base e , since we defined the number e to be the

number at which the function $\ln(x)$ takes the value 1. So we needed to have the natural logarithm available to us in the first place in order to say precisely what the number e is. Furthermore, by defining the natural logarithm function in the way we did, it is immediate that its derivative is $1/x$. This is the single most important property of this function.

4. SOME OTHER INTEGRALS

Recall again that the natural logarithm allows us to integrate $1/x$, *at least when $x > 0$* :

$$\begin{aligned} \text{For } x > 0, \quad \frac{d}{dx} \ln(x) &= \frac{1}{x} \\ \implies \int \frac{1}{x} dx &= \ln(x) + C \quad \text{for } x > 0 \end{aligned}$$

How do we integrate $1/x$ when x is negative. For instance, how can we find the area under $y = 1/x$ from $x = -5$ to $x = -2$? The following result deals with this question:

Theorem 4.1.

$$\int \frac{1}{x} dx = \ln|x| + C \quad \text{for } x \neq 0$$

Proof: We must show that

$$\frac{d}{dx} \ln|x| = \frac{1}{x} \quad \text{for } x \neq 0$$

We break this into two cases:

Case 1: $x > 0$: The result is already known, since $|x| = x$ when $x > 0$.

Case 2: $x < 0$: So $|x| = -x$ in this case. Thus

$$\frac{d}{dx} \ln|x| = \frac{d}{dx} \ln(-x) = \frac{1}{-x} \cdot -1 = \frac{1}{x}$$

We are now in a position to integrate the remaining trigonometric functions. Surprisingly, the answers involve the natural logarithm:

$$\begin{aligned}
\int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx \\
&= \int \frac{1}{\cos x} \sin x \, dx \\
&\quad [\text{Let } u = \cos x, \, du = -\sin x \, dx \,] \\
&= - \int \frac{1}{u} \, du \\
&= -\ln |u| + C \\
&= -\ln |\cos x| + C \\
&= \ln \left| \frac{1}{\cos x} \right| + C \\
&= \ln |\sec x| + C
\end{aligned}$$

Similarly

$$\int \cot x \, dx = \ln |\sin x| + C$$

Now

$$\int \sec x \, dx = \int \frac{1}{\cos x} \, dx$$

but does gets us nowhere.

Integrating $\sec x$ involves a trick pulled out of a hat:

Multiply and divide by $\sec x + \tan x$ (thus not changing the value of the integral):

$$\int \frac{\sec x(\tan x + \sec x)}{\tan x + \sec x} \, dx.$$

How does this help? Observe that

$$\frac{d}{dx}(\tan x + \sec x) = \sec^2 x + \sec x \tan x = \sec x(\sec x + \tan x).$$

Thus, making the substitution $u = \tan x + \sec x$, we get $du = \sec x(\tan x + \sec x) \, dx$ and the integral becomes

$$\int \frac{du}{u} = \ln |u| + C = \ln |\sec x + \tan x| + C.$$

The conclusion is:

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C$$

Similarly (check it!),

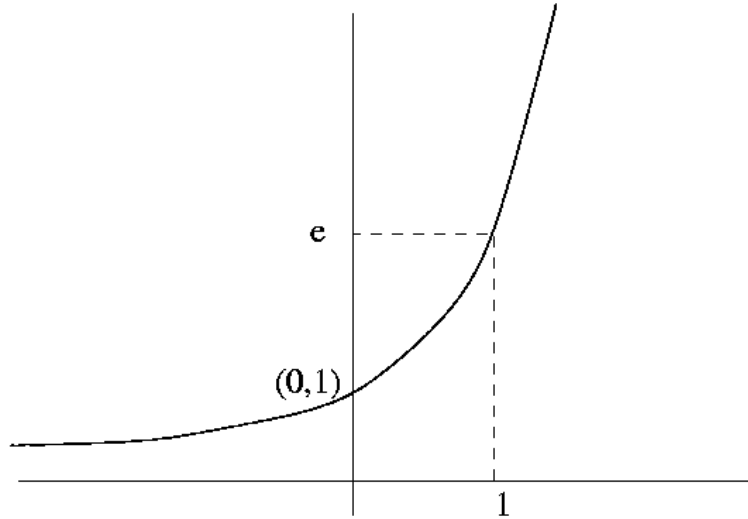
$$\int \csc x \, dx = -\ln |\csc x + \cot x| + C$$

5. THE NATURAL EXPONENTIAL FUNCTION

Definition 5.1. The (*natural*) *exponential function* is the function

$$\exp : \mathbb{R} \rightarrow \mathbb{R}, \quad \exp(x) = e^x$$

i.e., it is the exponential function to the base e .



Recall that for any positive base b ,

$$\log_b(b^x) = x \text{ for all } x \text{ and } b^{\log_b x} = x \text{ for } x > 0.$$

Thus in the particular case $b = e$, $e^{\log_e(x)} = x$ and $\log_e(e^x) = x$ so that

$$\exp(\ln x) = x \quad (x > 0) \text{ and } \ln(\exp x) = x \quad (x \in \mathbb{R})$$

Since there is a nice simple formula for the derivative of $\ln x$, It is natural to ask whether there is a similarly nice formula for the derivative of $\exp x$. The answer is yes. But to arrive at this formula we need to explore more closely the relationship between logarithms and exponential functions. They are examples of *inverse pairs of functions*.